

UNITARIZABILITY OF PREMODULAR CATEGORIES

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ABSTRACT. We study the unitarizability of premodular categories constructed from representations of quantum group at roots of unity. We introduce *Grothendieck unitarizability* as a natural generalization of unitarizability to classes of premodular categories with a common Grothendieck semiring. We obtain new results for quantum groups of Lie types F_4 and G_2 , and improve the previously obtained results for Lie types B and C .

1. INTRODUCTION

A unitary premodular category is a premodular category over \mathbb{C} whose morphisms spaces have been equipped with a positive-definite Hermitian form, compatible with the other structures (*i.e.* braiding, duality etc.). In particular for any object V the vector space $\text{End}(V)$ become a Hilbert space, and the braiding morphisms act by unitary operators. It is a difficult task, in general, to show that a given Hermitian premodular category is unitary (see [10] and [19]). However, one can study the weaker notion of *pseudo-unitarity*. A premodular category is pseudo-unitary if the categorical dimension of any object is a positive number (although this is not the most general definition, see Section 2 below). Pseudo-unitarity can often be checked directly, and since any unitary premodular category is automatically pseudo-unitary it is a useful statistic. Furthermore, there do not appear to be any examples of pseudo-unitary Hermitian premodular categories that are not unitarizable.

Unitary premodular categories have important applications in several areas of mathematics and physics. Firstly, they are a rich source of unitary representations of the braid group. Secondly, they can be used to construct II_1 von Neumann factors (see [19]). In case the categories are modular as well, one can construct 3-dimensional Topological Quantum Field Theories (TQFTs) (see [18]). They can also be regarded as algebraic models for exotic 2-dimensional physical (anyonic) systems [12] such as the quasi-particles in the Fractional Quantum Hall Effect. Recently these last two applications have been combined to form the basis for

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topological quantum computation [5]. This has lead to renewed interest in constructing and studying unitary premodular categories. Fortunately there are several well-known constructions of premodular categories, one of which is the subject of this article.

To any simple Lie algebra \mathfrak{g} and $q \in \mathbb{C}$ with q^2 a primitive ℓ th root of unity, one may associate a Hermitian premodular category $\mathcal{C}(\mathfrak{g}, q, \ell)$ over \mathbb{C} . The construction is well-known (see for example [2]): one begins with the category of finite-dimensional representations $\text{Rep}(U_q \mathfrak{g})$ of Lusztig's integral form of the quantum group $U_q \mathfrak{g}$, then passes to the (ribbon) subcategory \mathcal{T} of tilting modules (due to H. Andersen). The quotient of \mathcal{T} by the tensor ideal of negligible morphisms yielding the finite semisimple ribbon (=premodular) category $\mathcal{C}(\mathfrak{g}, q, \ell)$. Kirillov Jr. [10] defined an Hermitian structure on $\mathcal{C}(\mathfrak{g}, q, \ell)$, and conjectured that for the choice $q = e^{\pi i/\ell}$ the form is positive definite provided $2 \mid \ell$ if \mathfrak{g} is of Lie type B, C or F_4 and $3 \mid \ell$ if \mathfrak{g} is of Lie type G_2 . Subsequently, Wenzl [19] proved this conjecture, and we showed [13] that the hypothesis $2 \mid \ell$ is necessary for Lie types B and C for ℓ sufficiently large. In fact in [13] it is shown that unitarity fails in a much stronger sense: no Hermitian premodular category with the same Grothendieck semiring as $\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell)$ can even be pseudo-unitary when ℓ is odd with $4k + 3 \leq \ell$. This result was obtained by appealing to classification results found in [17].

In this paper we investigate several notions of unitarity for categories of the form $\mathcal{C}(\mathfrak{g}, q, \ell)$ where \mathfrak{g} is of Lie type B, C, F_4 and ℓ is odd or \mathfrak{g} is of Lie type G_2 and $3 \nmid \ell$. The basic result in these cases is that if ℓ is large enough none of the categories $\mathcal{C}(\mathfrak{g}, q, \ell)$ is pseudo-unitary for any choice of q . We also show that for some small values of ℓ , there *are* choices of q so that $\mathcal{C}(\mathfrak{g}, q, \ell)$ is pseudo-unitary. In addition we show that in some cases (with ℓ small) there are unitary premodular categories with the same Grothendieck semiring as $\mathcal{C}(\mathfrak{g}, q, \ell)$. However, these anomalies can be explained as “low-rank coincidences.” This completes the story for these types of categories, complementing the results in [19] and [13], sharpening the results of the latter.

Here is a more detailed description of the contents of this paper. In Section 2 we briefly describe the properties of premodular categories that are germane to our problem, introduce the new notion of Grothendieck unitarizability. and describe the premodular categories coming from quantum groups at roots of unity. In Section 3 we prove some general results for these categories and then prove our main results for Lie types G_2, F_4, B and C . In the Appendix we give an explicit equivalence between two quantum group categories of Lie types F_4 and E_8 respectively.

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2. BACKGROUND

Here we establish the setting in which we choose to work. For the most part we will be interested in Hermitian premodular categories. By a *premodular category* \mathcal{C} we mean an abelian semisimple \mathbb{C} -linear ribbon category with finitely many isomorphism classes of simple objects (see [18] for a more detailed description). In other settings a premodular category is called a *ribbon fusion category*. A *Hermitian* premodular category is equipped with an involution $f \mapsto \bar{f}$ on morphisms so that the pairing $(f, g) \rightarrow \text{tr}_{\mathcal{C}}(f\bar{g})$ is Hermitian and non-degenerate. One also requires the involution to restrict to ordinary complex conjugation on the base field \mathbb{C} .

2.1. Grothendieck Equivalence. For any semisimple braided tensor category \mathcal{C} the tensor product decomposition rules are described by the **Grothendieck ring** $Gr(\mathcal{C})$: unital based ring (see [11]) with basis consisting of the isomorphism classes of simple objects $[X]$ with $[X] \cdot [Y] = [X \otimes Y]$, and $[X] + [Y] = [X \oplus Y]$, with unit $[\mathbb{1}]$ where $\mathbb{1}$ is the unit object. We will sometimes abuse notation and write X instead of $[X]$ for the elements of $Gr(\mathcal{C})$ as this will cause no confusion. If there are finitely many simple classes $[X_i]$, $0 \leq i \leq n-1$ we say \mathcal{C} has *rank* n , and $Gr(\mathcal{C})$ can be identified with the finite rank unital based ring $\mathbb{Z}[x_0, x_1, \dots, x_{n-1}]/\mathcal{I}$ where the ideal \mathcal{I} is generated by $\{x_i x_j - \sum_k N_{i,j}^k x_k\}$ given that $X_i \otimes X_j \cong \sum_k N_{i,j}^k X_k$. The structure constants $N_{i,j}^k$ satisfy a number of useful symmetries. If we denote by i^* the index corresponding to the object X_i^* dual to X_i then we have:

$$(2.1) \quad N_{i,j}^k = N_{j,i}^k = N_{i,k}^{j^*} = N_{i^*,j^*}^{k^*} \quad \text{and} \quad N_{i,j}^0 = \delta_{j,j^*}$$

The Grothendieck ring has a useful matrix representation given by $X_i \rightarrow N_i$ where N_i is the $n \times n$ matrix with (k, j) entry equal to $N_{i,j}^k$. By the above symmetries one sees that $N_{i^*} = N_i^T$ where T is the transpose operator. If $i = i^*$ for all i (i.e. \mathcal{C} is a *self-dual* category), then the $N_{i,j}^k$ are totally symmetric in their indices.

The Grothendieck ring plays an important role in the theory of braided tensor categories due to a result that is known as *generalized Ocneanu rigidity* (see [3][Section 2.7]):

Proposition 2.2. *For a fixed unital based ring A , there are at most finitely many inequivalent premodular categories \mathcal{C} with $Gr(\mathcal{C}) \cong A$.*

We should point out that this theorem is stated in [3] for braided fusion categories, but any such category has at most finitely many ribbon structures (see, for example, [8][Lemma XIV.3.4]).

Motivated by this theorem we make the following definition:

Definition 2.3. Let \mathcal{C} and \mathcal{F} be two premodular categories. We say \mathcal{C} and \mathcal{F} are **Grothendieck equivalent**, and write $\mathcal{C} \stackrel{Gr}{\cong} \mathcal{F}$ if $Gr(\mathcal{C}) \cong Gr(\mathcal{F})$ as unital based rings.

An isomorphism between unital based rings is a bijection between the bases compatible with the fusion rules, so to show that two categories are Grothendieck equivalent it is enough to show that there is a bijection between the given bases that preserves the structure constants N_{ij}^k .

We have the following generalized notion of dimension:

Definition 2.4. A **dimension function** for a braided fusion category \mathcal{C} is a ring homomorphism $\phi : Gr(\mathcal{C}) \rightarrow \mathbb{C}$.

In any ribbon category \mathcal{C} there is a canonical dimension function $\dim_{\mathcal{C}}$ defined for any object X as the categorical trace of the identity morphism $Id_X \in \text{End}(X)$. It is well-known that $\dim_{\mathcal{C}}(X) = \dim_{\mathcal{C}}(X^*)$ for any object X , and that $\dim_{\mathcal{C}}(X) \in \mathbb{R}$. Notice that if ϕ is a dimension function for \mathcal{C} then the vector $\mathbf{f} = \sum_i \phi(X_i) \mathbf{e}_i$ is an eigenvector for each matrix N_i with eigenvalue $\phi(X_{i^*})$ (where \mathbf{e}_i is a standard basis vector for \mathbb{R}^n). This follows from the fact that $N_i \mathbf{e}_j = \sum_k N_{i,j}^k \mathbf{e}_k$ and $\phi(X_{i^*})\phi(X_k) = \sum_j N_{i^*,k}^j \phi(X_j)$ and the calculation:

$$N_i \mathbf{f} = \sum_j \phi(X_j) \sum_k N_{i,j}^k \mathbf{e}_k = \sum_k \left(\sum_j N_{i^*,k}^j \phi(X_j) \right) \mathbf{e}_k = \sum_k \phi(X_{i^*}) \phi(X_k) \mathbf{e}_k = \phi(X_{i^*}) \mathbf{f}.$$

Definition 2.5. If \mathcal{C} is a ribbon category, the **global dimension** $\dim(\mathcal{C})$ of the category is the sum of the squares of the categorical dimensions of the simple objects.

In more general settings the global dimension is defined differently (see [3]), but for ribbon (or more generally, spherical) categories the definition we give is equivalent to the standard one.

Definition 2.6. The **Frobenius-Perron dimension** of an object X in \mathcal{C} is the largest real eigenvalue of the matrix N_X of multiplication by X in the the Grothendieck ring $Gr(\mathcal{C})$, and is denoted $\text{FPdim}(X)$. $\text{FPdim}(\mathcal{C})$ is defined to be the sum of the squares of $\text{FPdim}(X_i)$ for simple classes X_i . A category is called **pseudo-unitary** if $\dim(\mathcal{C}) = \text{FPdim}(\mathcal{C})$.

From [3] we extract some important properties of FPdim :

- (1) $\text{FPdim}(X) > 0$ for any object (in particular $\text{FPdim}(X) \in \mathbb{R}$).
- (2) $X \mapsto \text{FPdim}(X)$ defines a dimension function.
- (3) $\text{FPdim}(X)$ is the unique dimension function with $\text{FPdim}(X) > 0$ for all objects X .

It also follows from the argument above that $\text{FPdim}(X) = \text{FPdim}(X^*)$, since the largest eigenvalues of M and M^T are the same.

Pseudo-unitarity does not immediately imply that $\dim(X_i) = \text{FPdim}(X_i)$ for each simple object. It is true (see [3][Prop. 8.3]) that if \mathcal{C} is any pseudo-unitary fusion category then there is a unique *spherical* structure on \mathcal{C} so that \dim and FPdim coincide, but if \mathcal{C} is a pseudo-unitary ribbon category there may not be

a *ribbon* structure on \mathcal{C} satisfying $\dim(X_i) = \text{FPdim}(X_i)$ for all simple X_i . In general we have the following:

Lemma 2.7. *Suppose \mathcal{C} is a pseudo-unitary premodular category. Then*

- (a) $\dim(X_i) = \pm \text{FPdim}(X_i)$ and
- (b) if X_i is a subobject of $X_j \otimes X_j^*$ for some simple X_j then $\dim(X_i) = \text{FPdim}(X_i)$.

Proof. Part (a) follows from

$$\dim(\mathcal{C}) = \sum_i (\dim(X_i))^2 = \sum_i (\text{FPdim}(X_i))^2 = \text{FPdim}(\mathcal{C})$$

and $|\dim(X_i)| \leq \text{FPdim}(X_i)$ (by the Perron-Frobenius theorem). If $\dim(X_i) < 0$ for some simple X_i then $\mathcal{C}_0 := \{X_i : \dim(X_i) = \text{FPdim}(X_i)\}$ and $\mathcal{C}_1 := \{X_i : \dim(X_i) = -\text{FPdim}(X_i)\}$ are (non-empty) bases for a \mathbb{Z}_2 -grading on $Gr(\mathcal{C})$. Since both $X_i \rightarrow \dim(X_i)$ and $X_i \rightarrow \text{FPdim}(X_i)$ give rise to characters of $Gr(\mathcal{C})$ it is clear that this grading is well-defined. Any simple subobject of $X_i \otimes X_i^*$ is in \mathcal{C}_0 since $\dim(X_i) = \dim(X_i^*)$, giving (b). \square

Finally, we give the following generalization of unitarizability:

Definition 2.8. Let \mathcal{C} be any premodular category, and $[\mathcal{C}]$ its Grothendieck equivalence class. We say \mathcal{C} is **Grothendieck unitarizable** if there is a unitary premodular category \mathcal{F} with $\mathcal{F} \in [\mathcal{C}]$.

Remark 2.9. The original form of Ocneanu rigidity states that there are finitely many *fusion* categories sharing a common Grothendieck semiring. So all of the notions described here work equally well for fusion categories. However we are interested in fusion categories that are at least braided so we have made our definitions in this more limited setting.

2.2. Quantum groups at roots of unity. In order to define and work with the categories $\mathcal{C}(\mathfrak{g}, q, \ell)$, we will need some (standard) notation and definitions. We make no attempt to be entirely self-contained, referring the reader to the survey paper [14] and the texts [2] and [18] for full details.

Let \mathfrak{g} be a simple Lie algebra, Φ its root system (resp. coroot system), with root basis Π and positive roots Φ_+ . We denote by $\check{\Phi}, \check{\Pi}$ and $\check{\Phi}_+$ the corresponding sets of coroots. Let $\langle \cdot, \cdot \rangle$ be the symmetric bilinear form on $\mathbb{R}\Phi$ normalized so that $\langle \alpha, \alpha \rangle = 2$ for *short* roots. We embed $\check{\Phi}$ and Φ in the same vector space by means of the identification $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. Observe that if α is a short root, $\check{\alpha} = \alpha$ under this identification. Denote by θ the highest root and by θ_s the highest short root. Define $m = \frac{\langle \beta, \beta \rangle}{\langle \alpha, \alpha \rangle}$ where β is a long root and α is a short root, so that $m = 1$ for Lie types A, D and E_N ($N = 6, 7$ or 8), $m = 2$ for Lie types B, C and F_4 and $m = 3$ for Lie type G_2 . Denote by ρ the half-sum of the positive roots, P

the set of weights, and P_+ the set of dominant weights labeling the irreducible highest-weight representations of \mathfrak{g} .

The **Weyl group** of \mathfrak{g} is the (finite) group W generated by the $|\Pi|$ *fundamental reflections* defined on $\mathbb{R}\Phi$ for each $\alpha_i \in \Pi$ by $s_i(\lambda) = \lambda - \langle \lambda, \check{\alpha}_i \rangle \alpha_i$. The **affine Weyl group** W_ℓ is generated by the fundamental reflections s_i and an additional translation operator T_ℓ . If $m \mid \ell$, then T_ℓ is the translation by $\ell\check{\theta}$ while if $m \nmid \ell$ T_ℓ is the translation by $\ell\check{\theta}_s = \ell\theta_s$. Often we will need to use the “dot” action of the (affine) Weyl group defined by $w.\lambda = w(\lambda + \rho) - \rho$. There is a well-defined multiplicative sign function ε on W_ℓ given by $\varepsilon(s_i) = -1$, $\varepsilon(T_\ell) = 1$ and extended to products.

With these definitions we can describe the Hermitian ribbon category $\mathcal{C}(\mathfrak{g}, q, \ell)$. The simple objects X_λ are labeled by the intersection C_ℓ of the set of dominant weights P_+ with the fundamental domain of the dot action of W_ℓ containing 0. Explicitly, we have:

$$C_\ell := \begin{cases} \{\lambda \in P_+ : \langle \lambda + \rho, \theta \rangle < \ell\} & \text{if } m \mid \ell \\ \{\lambda \in P_+ : \langle \lambda + \rho, \theta_s \rangle < \ell\} & \text{if } m \nmid \ell \end{cases}$$

The tensor product decomposition rules for $\mathcal{C}(\mathfrak{g}, q, \ell)$ are W_ℓ -antisymmetrizations of the tensor product decomposition rules for $\text{Rep}(U_q\mathfrak{g})$ with q not a root of unity or, equivalently, of $\text{Rep}(\mathfrak{g})$. Define $m_{\lambda\mu}^\nu = \dim \text{Hom}_{\mathfrak{g}}(V_\nu, V_\lambda \otimes V_\mu)$, *i.e.* the multiplicity of the irreducible Weyl module V_ν (over \mathfrak{g}) in the decomposition of $V_\lambda \otimes V_\mu$ into irreducible Weyl modules. Similarly, denote by $N_{\lambda\mu}^\nu$ the multiplicity of the object X_ν in $X_\lambda \otimes X_\mu$. Then we have the formula (see [1] and [15]):

$$(2.10) \quad N_{\lambda\mu}^\nu = \sum_{\substack{w \in W_\ell \\ w.\nu \in P_+}} \varepsilon(w) m_{\lambda\mu}^{w.\nu}$$

Having chosen some ordering on the set C_ℓ the matrices N_λ give a faithful representation of $Gr(\mathcal{C}(\mathfrak{g}, q, \ell))$ by extending $X_\lambda \rightarrow N_\lambda$ to all objects.

Remark 2.11. Observe that the structure constants $N_{\lambda\mu}^\nu$ depend only on ℓ , not on the particular choice of q with q^2 a primitive ℓ th root of unity. This implies that the class of categories $\mathfrak{C}(\mathfrak{g}, \ell) := \{\mathcal{C}(\mathfrak{g}, q, \ell) : q = e^{2\pi i/\ell}, \gcd(z, \ell)\}$ are Grothendieck equivalent. It follows from the strong form of Ocneanu rigidity (Proposition 2.2) that there are finitely many inequivalent premodular categories in $\mathfrak{C}(\mathfrak{g}, \ell)$.

The dimension function for $\mathcal{C}(\mathfrak{g}, q, \ell)$ is computed (for simple objects) by the formula:

$$(2.12) \quad \dim_q(X_\lambda) = \prod_{\alpha \in \Phi_+} \frac{[\langle \lambda + \rho, \alpha \rangle]}{[\langle \rho, \alpha \rangle]}$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ is the usual q -number. The categorical dimension is obtained by linear extension of this formula to direct sums of simple objects.

Proposition 2.13 (Wenzl). *Suppose $m \mid \ell$, and set $q = e^{\pi i/\ell}$. Then \dim_q is the Frobenius-Perron dimension for all of the Grothendieck equivalent categories in $\mathfrak{C}(\mathfrak{g}, \ell)$.*

3. MAIN RESULTS

We can now give the main results of the paper. We first prove some general results and then apply them to categories of Lie types G_2 , F_4 , B and C . In what follows the categories of Lie types G_2 and F_4 are always modular, while the categories of type B and C may be only premodular.

3.1. General results. In [13] the FPdim function was determined for categories $\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell)$, with ℓ odd. The proof found there contained some Lie-type- B -specific *ad-hoc* arguments that do not extend to other Lie types. The following generalizes that result:

Theorem 3.1. *Assume that $m \nmid \ell$ or $m = 1$, and define*

$$(3.2) \quad d_\lambda(q) := \prod_{\check{\alpha} \in \check{\Phi}_+} \frac{[\langle \lambda + \rho, \check{\alpha} \rangle]}{[\langle \rho, \check{\alpha} \rangle]}$$

where q is a formal variable. Then

$$d_\lambda(e^{\pi i/\ell}) = \text{FPdim}(X_\lambda)$$

is the Frobenius-Perron dimension for all of the Grothendieck equivalent categories in $\mathfrak{C}(\mathfrak{g}, \ell)$.

Proof. First observe that $d_\lambda(e^{\pi i/\ell})$ is positive since $0 < \langle \lambda + \rho, \check{\alpha} \rangle \leq \langle \lambda + \rho, \check{\theta}_s \rangle = \langle \lambda + \rho, \theta_s \rangle < \ell$ for any positive coroot $\check{\alpha}$, so that evaluating any factor $[\langle \lambda + \rho, \check{\alpha} \rangle]$ of the numerator of $d_\lambda(q)$ at $e^{\pi i/\ell}$ gives

$$\sin(\langle \lambda + \rho, \check{\alpha} \rangle \pi / \ell) > 0$$

and similarly for any factor of the denominator. Since the Frobenius-Perron dimension is unique among positive dimension functions, it suffices to show that $d_\lambda(e^{\pi i/\ell})$ is indeed a dimension function for $\mathcal{C}(\mathfrak{g}, q, \ell)$. This is achieved first by showing that $d_\lambda(q)$ is a dimension function for $\text{Rep}(U_q \mathfrak{g})$, and then showing that $d_\lambda(e^{\pi i/\ell})$ exhibits the requisite W_ℓ -antisymmetry to derive the result for $\mathcal{C}(\mathfrak{g}, q, \ell) \in \mathfrak{C}(\mathfrak{g}, \ell)$.

Applying the Weyl denominator de-factorization formula to the coroot system with normalized W -invariant form $\langle \cdot, \cdot \rangle' = m \langle \cdot, \cdot \rangle$ one obtains

$$d_\lambda(q) = \frac{\sum_{w \in W} \varepsilon(w) q^{\langle w(\lambda + \rho), 2\check{\rho} \rangle}}{\sum_{w \in W} \varepsilon(w) q^{\langle w(\rho), 2\check{\rho} \rangle}}$$

where $\check{\rho}$ is half the sum of the positive coroots. But this is precisely the character χ_λ of the $U_q\mathfrak{g}$ -module V_λ evaluated at a particular element (see [10][Section 3] for example). Since $\chi_\lambda\chi_\mu = \sum_\nu m_{\lambda\mu}^\nu\chi_\nu$, we have that $d_\lambda(q)$ is a dimension function for $\text{Rep}(U_q\mathfrak{g})$. To show that

$$d_\lambda(e^{\pi i/\ell})d_\mu(e^{\pi i/\ell}) = \sum_\nu N_{\lambda\mu}^\nu d_\nu(e^{\pi i/\ell})$$

we need only show that $d_{w.\lambda}(e^{\pi i/\ell}) = \varepsilon(w)d_\lambda(e^{\pi i/\ell})$ for any $w \in W_\ell$. It suffices to check that $d_\lambda(e^{\pi i/\ell})$ is invariant under the dot action of the generator T_ℓ (since $\varepsilon(T_\ell) = 1$) and anti-invariant under the dot action of the generators s_i (with $\varepsilon(s_i) = -1$). For this we compute:

$$\langle w(s_i.\lambda + \rho), 2\check{\rho} \rangle = \langle w(s_i(\lambda + \rho) - \rho + \rho), 2\check{\rho} \rangle = \langle ws_i(\lambda + \rho), 2\check{\rho} \rangle$$

and since $\varepsilon(ws_i) = -\varepsilon(w)$ the anti-symmetry with respect to s_i follows by reindexing the sum. Similarly for T_ℓ we compute (recalling that T_ℓ is the translation by $\ell\check{\theta}_s = \ell\theta_s$ in this case):

$$\begin{aligned} \langle w(T_\ell.\lambda + \rho), 2\check{\rho} \rangle &= \langle T_\ell(\lambda + \rho), w^{-1}(2\check{\rho}) \rangle = \langle \lambda + \rho + \ell\theta_s, w^{-1}(2\check{\rho}) \rangle = \\ &= \langle w(\lambda + \rho), 2\check{\rho} \rangle + \ell \langle w(\theta_s), 2\check{\rho} \rangle \end{aligned}$$

and since $\ell \langle w(\theta_s), 2\check{\rho} \rangle$ is an *even* multiple of ℓ (as $\langle w(\theta_s), \check{\rho} \rangle \in \mathbb{Z}$), $q^{\ell \langle w(\theta_s), 2\check{\rho} \rangle} = 1$. Thus $d_\lambda(e^{\pi i/\ell})$ is invariant under the dot action of T_ℓ . This completes the proof that $d_\lambda(e^{\pi i/\ell}) = \text{FPdim}(X_\lambda)$. \square

Remark 3.3. The statement of Theorem 3.2 is false if $m \mid \ell$ and $m \neq 1$. The proof fails because the translation T_ℓ in those cases is by $\ell\check{\theta} = \frac{\ell}{m}\theta$, so that $\ell \langle w(\check{\theta}), 2\check{\rho} \rangle$ is *not* always an even multiple of ℓ .

3.2. Lie Type G_2 . The category $\mathcal{C}(\mathfrak{g}_2, q, 7)$ is the trivial rank 1 category for any choice of q , and the rank of $\mathcal{C}(\mathfrak{g}_2, q, \ell)$ for $3 \nmid \ell$ can be computed via the generating function $\frac{1}{(1-x)(1-x^2)(1-x^3)}$ from [14]. If we label the short fundamental weight (corresponding to the 7-dimensional rep of \mathfrak{g}_2) by Λ_1 and the long fundamental weight by Λ_2 we obtain the following formula for \dim_q of the simple object X_λ labeled by weight $\lambda = \lambda_1\Lambda_1 + \lambda_2\Lambda_2$:

$$\frac{[\lambda_1 + 1][3(\lambda_2 + 1)][\lambda_1 + 3\lambda_2 + 4][3(\lambda_1 + \lambda_2 + 2)][2\lambda_1 + 3\lambda_2 + 5][3(\lambda_1 + 2\lambda_2 + 3)]}{[1][3][4][5][6][9]}.$$

The formula for $\text{FPdim}(X_\lambda)$ is similar to the above except one must cancel the factors of 3 in the q -numbers of the form $[3n]$ in the numerator and denominator, and then evaluate at $e^{\pi i/\ell}$.

For ℓ sufficiently large $\mathcal{C}(\mathfrak{g}_2, q, \ell)$ is not (pseudo-)unitary for any choice of q , but for smaller values of ℓ one obtains some unitarity results. In particular, we have the following:

Theorem 3.4. *Let ℓ be an integer with $\ell > 7$ and $3 \nmid \ell$, and $q = e^{\pi iz/\ell}$ with $\gcd(z, \ell) = 1$. Then*

- (a) *if $\ell \geq 16$, $\mathcal{C}(\mathfrak{g}_2, q, \ell)$ is not pseudo-unitary for any choice of q ,*
- (b) *the categories $\mathcal{C}(\mathfrak{g}_2, e^{\pi i 2/11}, 11)$, $\mathcal{C}(\mathfrak{g}_2, e^{\pi i 3/13}, 13)$ and $\mathcal{C}(\mathfrak{g}_2, e^{\pi i 3/14}, 14)$ are pseudo-unitary, and*
- (c) *$\mathcal{C}(\mathfrak{g}_2, q, 8)$, $\mathcal{C}(\mathfrak{g}_2, q, 10)$ and $\mathcal{C}(\mathfrak{g}_2, q, 11)$ are Grothendieck unitarizable.*

Proof. For part (a) fix $\ell \geq 16$. Observe that X_{Λ_1} appears as a subobject of $X_{\Lambda_1}^{\otimes 2}$, so that if $\mathcal{C}(\mathfrak{g}_2, q, \ell)$ were pseudo-unitary we would have $\dim_q(X_{\Lambda_1}) = \text{FPdim}(X_{\Lambda_1})$ by Lemma 2.7. Thus for part (a) it is enough to show that $\dim_q \neq \text{FPdim}$ for any choice of q , which can be accomplished by showing $\dim_q(X_{\Lambda_1}) < \text{FPdim}(X_{\Lambda_1})$. Comparing formulas (3.2) and (2.12), we see that this amounts to showing

$$(3.5) \quad \frac{\sin(7z\pi/\ell) \sin(2z\pi/\ell) \sin(12z\pi/\ell)}{\sin(z\pi/\ell) \sin(4z\pi/\ell) \sin(6z\pi/\ell)} < \frac{\sin(7\pi/\ell)}{\sin(\pi/\ell)}$$

for any $1 \leq z \leq \ell - 1$ with $\gcd(z, \ell) = 1$. By invariance of $\dim_q(X_{\Lambda_1})$ we may assume further that $1 \leq z \leq \frac{\ell-1}{2}$. For notational convenience let us introduce the new variable $t = z\pi/\ell$ so that $0 < t < \pi/2$. Standard trigonometric identities yield

$$\dim_q(X_{\Lambda_1}) = \frac{\sin(7t) \sin(2t) \sin(12t)}{\sin(t) \sin(4t) \sin(6t)} = \frac{\sin(7t)}{\sin(t)} (4 \cos^2(2t) - 3).$$

Since $-3 \leq 4 \cos^2(2t) - 3 \leq 1$, we consider two cases: (I) $\frac{\sin(7t)}{\sin(t)} < 0$ and (II) $\frac{\sin(7t)}{\sin(t)} > 0$.

In case (I) we compute the minimum of $\frac{\sin(7t)}{\sin(t)}$ to be $-\frac{7+14\sqrt{7}}{27} > -1.632$. So in case (I) we have that $\dim_q(X_{\Lambda_1}) < (-3)(-1.632) < 4.9$. Next notice that $\text{FPdim}(X_{\Lambda_1})$ is an increasing function of ℓ , so we have

$$5.02 < \frac{\sin(7\pi/16)}{\sin(\pi/16)} \leq \text{FPdim}(X_{\Lambda_1})$$

hence in case (I) we see that (3.5) holds.

For case (II) we first observe that (3.5) holds for $z = 1$, since $4 \cos^2(2\pi/\ell) - 3 \neq 1$, reducing the problem to showing

$$\frac{\sin(7z\pi/\ell)}{\sin(z\pi/\ell)} < \frac{\sin(7\pi/\ell)}{\sin(\pi/\ell)}$$

on $2 \leq z \leq \frac{\ell-1}{2}$. One sees that the function $\frac{\sin(7t)}{\sin(t)}$ is decreasing on the interval $0 < t < 1/2$, corresponding to $0 < z < \frac{\ell}{2\pi}$, so that $\frac{\sin(7z\pi/\ell)}{\sin(z\pi/\ell)} < \frac{\sin(7\pi/\ell)}{\sin(\pi/\ell)}$ certainly holds on the interval $(1, \frac{\ell}{2\pi})$. Moreover, on $1/2 \leq t \leq \pi/2$, we have $\frac{\sin(7t)}{\sin(t)} < 2 < \frac{\sin(7\pi/\ell)}{\sin(\pi/\ell)}$ so that (3.5) also holds in case (II), completing the proof of (a).

The proof of (b) is simply a computation: one uses formula 2.12 for \dim_q evaluated at the given value of q to check positivity on the (5, 8 and 10) simple objects for the given categories corresponding to $\ell = 11, 13$ and 14.

To prove part (c) we explicitly determine the fusion rules for the given categories by computing the classical multiplicities $m_{\lambda\mu}^\nu$ (for instance by using Stembridge's Maple packages coxeter/weyl [16]) and applying formula (2.10). Then we recognize the fusion rules as those of well-known unitary premodular categories obtained from the quantum group $U_q \mathfrak{sl}_2$. In particular, we have:

- (1) $\mathcal{C}(\mathfrak{g}_2, q, 8) \stackrel{Gr}{=} \mathcal{C}(\mathfrak{sl}_2, e^{\pi i/3}, 3),$
- (2) $\mathcal{C}(\mathfrak{g}_2, q, 10) \stackrel{Gr}{=} \mathbb{Z}(\mathcal{C}(\mathfrak{sl}_2, e^{\pi i/5}, 5)) \boxtimes \mathbb{Z}(\mathcal{C}(\mathfrak{sl}_2, e^{\pi i/5}, 5))$ and
- (3) $\mathcal{C}(\mathfrak{g}_2, q, 11) \stackrel{Gr}{=} \mathbb{Z}(\mathcal{C}(\mathfrak{sl}_2, e^{\pi i/11}, 11))$

where $\mathbb{Z}(\mathcal{C}(\mathfrak{sl}_2, q, \ell))$ is the subcategory generated by the simple objects with even labels X_0, X_2, \dots , *i.e.* with categorical dimensions $1 = [1], [3], [5], \dots$, and \boxtimes is the tensor product of \mathbb{C} -linear categories. Thus these categories are Grothendieck unitarizable. \square

3.3. Lie type F_4 . The category $\mathcal{C}(\mathfrak{f}_4, q, \ell)$ with ℓ odd is trivial for $\ell = 13$ and has rank 4, 10, 21, 39... for $\ell = 15, 17, 19, 21...$ as can be obtained from the generating function $[(1-x)(1-x^2)^2(1-x^3)(1-x^4)]^{-1}$ (see [14]). We label the four fundamental weights by Λ_i , $1 \leq i \leq 4$ where Λ_1 corresponds to the 26-dimensional representation of \mathfrak{f}_4 .

As in the Lie type G_2 case, when ℓ is large enough, the categories $\mathcal{C}(\mathfrak{f}_4, q, \ell)$ are not pseudo-unitary, while for small values of ℓ one does obtain some forms of unitarity. Specifically we have:

Theorem 3.6. *Let ℓ be an integer with $\ell > 13$ and $2 \nmid \ell$, and $q = e^{\pi iz/\ell}$ with $\gcd(z, \ell) = 1$. Then*

- (a) *if $\ell \geq 19$, $\mathcal{C}(\mathfrak{f}_4, q, \ell)$ is not pseudo-unitary for any choice of q ,*
- (b) *$\mathcal{C}(\mathfrak{f}_4, e^{3\pi i/17}, 17)$ is pseudo-unitary and*
- (c) *$\mathcal{C}(\mathfrak{f}_4, q, 15)$ and $\mathcal{C}(\mathfrak{f}_4, q, 17)$ are Grothendieck unitarizable.*

Proof. The proof of (a) follows the same strategy as in the Lie type G_2 case. Once again X_{Λ_1} appears as a subobject of its own second tensor power, so that pseudo-unitarity would imply $\dim_q(X_{\Lambda_1}) = \text{FPdim}(X_{\Lambda_1})$. One reduces to showing that

$$(3.7) \quad \frac{\sin(3z\pi/\ell) \sin(8z\pi/\ell) \sin(13z\pi/\ell) \sin(18z\pi/\ell)}{\sin(z\pi/\ell) \sin(4z\pi/\ell) \sin(6z\pi/\ell) \sin(9z\pi/\ell)} < \frac{\sin(8\pi/\ell) \sin(13\pi/\ell)}{\sin(\pi/\ell) \sin(4\pi/\ell)}$$

for $\ell \geq 19$ and $1 \leq z \leq \frac{\ell-1}{2}$. We omit the details, pausing to note that the relevant trigonometric identity in terms of $t = z\pi/\ell$ is

$$\frac{\sin(18t) \sin(3t)}{\sin(6t) \sin(9t)} = 4 \cos^2(3t) - 3.$$

For part (b) one simply computes \dim_q 10 simple objects corresponding to $\ell = 17$ and verifies that they are positive.

For part (c) we have the following explicit Grothendieck equivalences with unitary premodular categories:

- (1) $\mathcal{C}(\mathfrak{f}_4, q, 15) \stackrel{Gr}{=} \mathbb{Z}(\mathcal{C}(\mathfrak{sl}_2, e^{\pi i/5}, 5)) \boxtimes \mathbb{Z}(\mathcal{C}(\mathfrak{sl}_2, e^{\pi i/5}, 5))$
- (2) $\mathcal{C}(\mathfrak{f}_4, q, 17) \stackrel{Gr}{=} \mathcal{C}(\mathfrak{e}_8, e^{\pi i/34}, 34)$.

The case $\ell = 15$ is fairly simple to verify, while the case $\ell = 17$ is handled in the Appendix. \square

3.4. Lie Types B and C . For Lie types B and C we have a somewhat stronger statement. For these types, we can combine the classification theorem in [17] and the results in [13] to show that any braided fusion category that is Grothendieck equivalent to $\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell)$ for some q with ℓ odd has the same dimension function for some (possibly different) choice of q . Note that these categories are trivial for $\ell = 2k + 1$. We have the following theorem, part (a) of which is a sharpening of [13][Theorem 7.1]:

Theorem 3.8. *Let ℓ be an odd integer with $2k + 1 < \ell$, and $q = e^{\pi iz/\ell}$ with $\gcd(z, \ell) = 1$. Then*

- (a) *if $2k + 5 \leq \ell$, neither $\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell)$ nor $\mathcal{C}(\mathfrak{sp}_{\ell-2k-1}, q, \ell)$ is pseudo-unitary or Grothendieck unitarizable for any choice of q , and*
- (b) *if $\ell = 2k + 3$ then both $\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell)$ and $\mathcal{C}(\mathfrak{sp}_{\ell-2k-1}, q, \ell)$ are Grothendieck unitarizable.*

Proof. First one reduces to considering only $\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell)$ using a rank-level duality result: $\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell) \stackrel{Gr}{=} \mathcal{C}(\mathfrak{sp}_{\ell-2k-1}, q, \ell)$ established in [13][Corollary 6.6].

The proof of (a) relies upon establishing that $\dim_q(X_{\Lambda_1}) \neq \text{FPdim}(X_{\Lambda_1})$ where Λ_1 is the fundamental weight corresponding to the vector representation of \mathfrak{so}_{2k+1} . Since X_{Λ_1} is a subobject of $X_{\Lambda_k} \otimes X_{\Lambda_k}^*$ (where X_{Λ_k} is the object corresponding to the fundamental spin representation of \mathfrak{so}_{2k+1}) this is a sufficient condition by Lemma 2.7. This amounts to showing that:

$$(3.9) \quad \frac{\sin((2k+1)z\pi/\ell) \sin(2(2k-1)z\pi/\ell)}{\sin(2z\pi/\ell) \sin((2k-1)z\pi/\ell)} < \frac{\sin((2k+1)\pi/\ell)}{\sin(\pi/\ell)}$$

for any $1 \leq z \leq \ell - 1$ with $\gcd(z, \ell) = 1$, which was done in [13][Lemma 7.2] for $4k + 3 \leq \ell$. A referee graciously provided us with the following proof of Inequality (3.9) in the general case:

First notice that the left-hand-side of Ineq. (3.9) can be written as $\frac{\sin(4kz\pi i/\ell)}{\sin(2z\pi/\ell)} + 1$. Using the identity:

$$\frac{\sin(s\alpha)}{\sin(\alpha)} + 1 = \frac{((-1)^s + 1)}{2} + 2 \sum_{j=0}^{\lfloor \frac{s-1}{2} \rfloor} \cos((s-1-2j)\alpha)$$

(where $\lfloor x \rfloor$ is the greatest integer function) one sees that Ineq. (3.9) is equivalent to:

$$(3.10) \quad \sum_{j=1}^k \cos(2(2j-1)z\pi/\ell) < \sum_{j=1}^k \cos(2j\pi/\ell).$$

Observe that each side of (3.10) is a sum of k distinct numbers of the form $\cos(2s\pi/\ell)$ for some integer s with $1 \leq s \leq (2k-1)z$. Since $\gcd(z, \ell) = 1$ and $2k-1 < \ell$ we see that $\ell \nmid s$, hence the largest of these numbers is $\cos(2\pi/\ell)$ followed by $\cos(4\pi/\ell)$ etc. so that the right-hand-side of (3.10) is the sum of the k largest numbers of this form. It follows that (3.10) holds provided it is not an equality. For this, first notice that (3.10) is an equality if and only if the sets of integers

$$S_1 = \{\pm 1, \pm 2, \dots, \pm k\} \quad \text{and} \quad S_2 = \{\pm z, \pm 3z, \dots, \pm(2k-1)z\}$$

are equal modulo ℓ (since $\cos(x)$ is an even function).

Suppose $S_1 = S_2$. If we imagine the set of integers modulo ℓ placed on a circle in the usual way, the subset S_2 has the following description: starting at $-(2k-1)z$, the remaining $2k-1$ elements of S_2 are obtained by moving counterclockwise in $2k-1$ steps of size $2z$. The subset S_1 consists of all of the inequivalent residues modulo ℓ except 0 and a gap $J = [k+1, \ell - (k+1)]$ of size $|J| = \ell - 2(k+1) + 1 \geq 4$. Consider two cases 1) $2z < |J|$ and 2) $2z \geq |J|$. In case 1) we arrive at an immediate contradiction since after some step of size $2z$ we will arrive at some element of S_1 in the gap $J = [k+1, \ell - (k+1)]$ (for example, one of $k+2z$ or $(k-1)+2z$ is such an element). In case 2), consider the three consecutive residues $\ell - k - 2z - 1, \ell - k - 2z - 2$ and $\ell - k - 2z - 3$. By hypothesis these three numbers are not in the gap, so they must be either in S_1 or equal to 0 and hence at least one of them must be of the form $(2i-1)z$ with $-k+1 \leq i < k$ (another possible value is $(2k-1)z$). Such a number shifted by $2z$ must remain in S_1 , but each of $\ell - k - 1, \ell - k - 2$ and $\ell - k - 3$ is in the gap (as $|J| \geq 4$), a contradiction. This proves (a).

For part (b), observe that if $\ell = 2k+3$ then $\ell - 2k - 1 = 2$, so that the rank-level duality and the isomorphism $\mathfrak{sp}_2 \cong \mathfrak{sl}_2$ implies that

$$\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell) \stackrel{Gr}{=} \mathcal{C}(\mathfrak{sp}_2, q, \ell) \stackrel{Gr}{=} \mathcal{C}(\mathfrak{sl}_2, q, \ell).$$

It is of course well-known that $\mathcal{C}(\mathfrak{sl}_2, q, \ell)$ is a unitary modular category for $q = e^{\pi i/\ell}$ (see e.g. [19]). Thus (b) is established. \square

3.5. Concluding remarks.

- (1) A natural question to ask is if Theorems 3.4(a) and 3.6(a) can be strengthened, replacing *pseudo-unitary* with *Grothendieck unitarizable* as in Theorem 3.8. Unfortunately a classification for fusion categories of Lie types E_N , F_4 or G_2 (in the spirit of the type $A-D$ classifications, see [9] and [17])

does not exist, making such a strengthening problematic (at least from our approach).

- (2) It is desirable to have more conceptual explanation of Theorems 3.4, 3.6 and 3.8. This could potentially be achieved from the observation that for $\mathcal{C}(\mathfrak{g}, q, \ell)$ with $m \mid \ell$ the corresponding categories are monoidally equivalent to the fusion categories of fixed level modules of the (untwisted) affine Kac-Moody algebras $\widehat{\mathfrak{g}}$ (see [4]), while the sets C_ℓ described above are related to the labeling sets of fixed level modules of twisted affine Kac-Moody algebras (see [7]) *for which no fusion product is known*. Moreover, A. Wasserman pointed out that the fusion category of level k modules for affine Kac-Moody algebras $\widehat{\mathfrak{g}}$ are naturally unitary.

APPENDIX

We claim that the categories $\mathcal{C}(\mathfrak{f}_4, q, 17)$ and $\mathcal{C}(\mathfrak{e}_8, q, 34)$ are Grothendieck equivalent. We will exhibit an isomorphism of unital based rings between the corresponding Grothendieck semirings. Let us label the four fundamental weights of $U_q \mathfrak{f}_4$ by Λ_i , $1 \leq i \leq 4$ with the label Λ_1 corresponding to the 26 dimensional representation of \mathfrak{f}_4 and the eight fundamental weights of $U_q \mathfrak{e}_8$ by λ_i , $1 \leq i \leq 8$ with λ_8 corresponding to the 248 dimensional representation of \mathfrak{e}_8 . The ten simple objects of $\mathcal{C}(\mathfrak{f}_4, q, 17)$ are labelled by

$$\mathcal{B} = \{0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, 2\Lambda_1, 2\Lambda_2, \Lambda_1 + \Lambda_4, \Lambda_1 + \Lambda_2, \Lambda_2 + \Lambda_4\},$$

while those of $\mathcal{C}(\mathfrak{e}_8, q, 34)$ are labelled by

$$\mathcal{D} = \{0, \lambda_8, \lambda_7, \lambda_6, 2\lambda_8, \lambda_1, 2\lambda_1, \lambda_1 + \lambda_8, \lambda_2, \lambda_3\}.$$

Let us use \mathcal{B} and \mathcal{D} to form ordered bases $\{X_i\}_{i=0}^9$ and $\{Y_i\}_{i=0}^9$ for the Grothendieck semirings $Gr(\mathcal{C}(\mathfrak{f}_4, q, 17))$ and $Gr(\mathcal{C}(\mathfrak{e}_8, q, 34))$ (so that $X_1 = X_{\Lambda_1}$ and $Y_1 = X_{\lambda_8}$, etc.). Using standard techniques (e.g. [16]) together with formula (2.10), one computes the fusion matrices N_{X_1} and N_{Y_1} relative to the given ordered bases, and finds that they are identical. For the reader's convenience we record:

$$N_{X_1} = N_{Y_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

By first writing the fusion rules for tensoring with X_1 as (commutative) polynomials in X_0, \dots, X_9 , a Gröbner basis computation (with monomials ordered by total degree) explicitly determines the remaining fusion rules. This proves that the two categories are Grothendieck equivalent.

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